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An exact solution for a diffusive flow in a porous medium

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It is shown that Yih's exact solution for the non-diffusive flow of a non-homogeneous fluid into a sink in a confined porous medium is equivalent to a class of diffusive flows with isopycnic lateral boundaries.

Yih (1961) investigated the non-diffusive two-dimensional flow of a non-homogeneous fluid in a porous medium. He determined the porous medium equivalent of his version of Long's equation for the flow of density-stratified fluids and applied it to solving the problem of the flow of a linearly-stratified fluid into a two-dimensional sink in a confined porous medium. In this note we show that there is an equivalence between the streamlines for the non-diffusive motion with constant viscosity and streamlines for the complete diffusive motion with isopycnic boundary conditions and that the density-distribution of the diffusive motion can be deduced from the solution given by Yih for the non-diffusive motion.

Fluid of constant dynamic viscosity μ containing a dissolved salt and in two-dimensional motion in a porous medium of intrinsic permeability k satisfies the equations

$$\mu\nabla^2\psi = gk\rho_x, \tag{1}$$

$$\sigma(\psi_z\rho_x - \psi_x\rho_z) = D\nabla^2\rho. \tag{2}$$

(See Yih (1961), List (1968) for nomenclature.) The first equation embodies Darcy's law and in fact states the conservation of vorticity; the second is derived from the mass conservation equation.

When the flow is non-diffusive the term $D\nabla^2\rho$ is not present and (2) then implies that the density ρ is a function of ψ so that (1) may be rewritten

$$\nabla^2\psi = \frac{gk}{\mu}\frac{d\rho}{d\psi}\frac{\partial\rho}{\partial x}, \tag{3}$$

(Yih's equation (15) for uniform viscosity) and $d\rho/d\psi$ is determined from the upstream conditions.

In the problem of confined flow of a linearly stratified fluid into a line sink as treated by Yih the upstream conditions (not explicitly stated by Yih) are

$$\rho = \rho_0 - \beta\rho_0(z/d) \quad (x \to -\infty), \tag{4}$$

$$\psi = Ud(z/d) \qquad (x \to -\infty), \tag{5}$$

giving $d\rho/d\psi = -\beta\rho_0/Ud$ so that the problem becomes specified by

$$\left.\begin{aligned} &\Psi_{\xi\xi}+\Psi_{\eta\eta}+B\Psi_{\xi} = 0,\\ &\Psi = 0 \quad \text{at} \quad \eta = 0 \quad \text{and} \quad \xi = 0 \quad (\eta < 1),\end{aligned}\right\} \tag{6}$$

$$\left.\begin{aligned} &\Psi = 1 \quad \text{at} \quad \eta = 1,\\ &\Psi = \eta, \quad \xi = -\infty.\end{aligned}\right\} \tag{7}$$

The solution is given by Yih as

$$\Psi = \eta+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin n\pi\eta \exp(\alpha\xi), \tag{8}$$

$$\alpha = \tfrac{1}{2}(-B+\sqrt{[B^2+4n^2\pi^2]}), \tag{9}$$

$$B = R = gk\beta\rho_0/\mu U. \tag{10}$$

When the motion is diffusive, it is apparent that a solution of (1) and (2) of the form $\rho = f(\psi)$ is no longer possible. However, in considering the same problem solved by Yih and maintaining the isopycnic conditions on lateral boundaries we determine the possibility of the existence of a solution of the form

$$\rho/\rho_0 = 1+a(z/d)+b(\psi/Ud), \tag{11}$$

where a and b are as yet undetermined constants.

For a solution of this form to satisfy conditions (4) and (5) at infinity it can easily be shown that

$$a+b = -\beta. \tag{12}$$

Furthermore, substituting (11) into both (1) and (2) we find (1) becomes

$$\nabla^2\psi = b\left(\frac{gk\rho_0}{\mu Ud}\right)\frac{\partial\psi}{\partial x} \tag{13}$$

and (2) becomes

$$\nabla^2\psi = -\left(\frac{a}{b}\right)\frac{\sigma U}{D}\frac{\partial\psi}{\partial x}. \tag{14}$$

Equations (13) and (14) will be exactly the same equation if

$$b\frac{gk\rho_0}{\mu Ud} = -\left(\frac{a}{b}\right)\frac{\sigma U}{D}, \tag{15}$$

thus giving another equation for a and b in addition to (12).

We solve for a and b and find that the diffusive problem is specified by identically the same equation and boundary conditions that specify the non-diffusive problem, namely (6) and (7) above, except that now

$$B = P[(1+2R/P)^{\frac{1}{2}}-1], \tag{16}$$

where $P = \sigma Ud/2D$ is a Peclet number; thus each solution of Yih's corresponds to a whole class of diffusive flow solutions. It can easily be shown that when $2R \ll P$ the parameter B above reduces to the value R it would have for a non-diffusive motion. The ratio of the two parameters R and P therefore gives a measure of the importance of diffusion in any given physical problem. Since it seems likely that in most physical situations $2R \ll P$ Yih's neglect of the diffusive terms would appear justified. However, it is still worthwhile considering the influence of the diffusive term. When this condition is not satisfied, the streamlines of the diffusive motion will still coincide with those for some non-diffusive motion at a lower value of R. The density distribution in the diffusive flow will, however, be modified from the non-diffusive distribution according to (11). To see the effect of diffusion on the streamline distribution we consider the case $R = 4\pi$. The following table gives the corresponding values of B for various diffusive motions.

R	P	B
4π	$\frac{1}{6}\pi$	π
4π	π	2π
4π	∞	4π

Thus a diffusive flow with $R = 4\pi$, $P = \frac{1}{6}\pi$ is equivalent to a non-diffusive motion with $R = \pi$, or with $P = \pi$ to a non-diffusive motion with $R = 2\pi$. The effect of diffusion is therefore displayed graphically in figures 1–3 in Yih (1961). Since P decreasing corresponds to an increasing diffusion coefficient it is readily apparent what the effect of this is. The diagrams show the flow tending more towards a uniform density type as P decreases, just as one would expect. A similar transformation for the axially symmetric problem does not appear possible.

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